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DYNAMICS OF FLEXIBLE SLIDING BEAMS— NON-LINEAR ANALYSIS PART II: TRANSIENT RESPONSE

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In this paper the axially rigid sliding beam problem undergoing small deformations is examined first, its governing equation and boundary conditions transformed to the fixed domain and the well known Galerkin's approach used to study the transient response of this problem. The results obtained are then compared with those in the literature. Subsequently the authors' approach is extended to the non-linear, axially inextensible sliding beams undergoing large amplitude vibrations and solve several examples to show the differences between the solutions obtained via linear and non-linear solvers.

1. INTRODUCTION

In part I, the equations of motion for the sliding beam problem were obtained by two formulations; by allowing the beam to move in a rigid and fixed channel and by holding the beam fixed and allowing the channel to move along the beam. Also by mapping the two formulations to the fixed domain, it was shown that the Eulerian–Lagrangian formulation (sliding beam formulation) and the full Lagrangian formulation (sliding channel formulation) lead to the same equation of motion. The fixed domain provides a very effective means for obtaining solutions for this complex problem (see also Vu-Quoc and Li [1]).

In this paper the inextensible sliding beams are considered and by using Galerkin's method, the transient response of the beam system obtained.

2. AXIALLY INEXTENSIBLE FLEXIBLE SLIDING BEAMS IN THE FIXED DOMAIN

In part I a general transformation for the equation of motion of an extensible sliding beam from a variable domain to a fixed domain was carried out. As an alternative, in the following a similar transformation on the Lagrangian of the system will be carried out but will restrict ones attention to the inextensible case.

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2.1. LAGRANGIAN IN THE FIXED DOMAIN

The Lagrangian for the axially inextensible sliding beam may be written as [2]

$$\mathscr{L}_0 = T^* - \Pi$$
$$= \int_0^{L(t)} \tfrac{1}{2}\rho A\left[\left(\frac{\partial x_1}{\partial t} + V\frac{\partial x_1}{\partial S}\right)^2 + \left(\frac{\partial x_2}{\partial t} + V\frac{\partial x_2}{\partial S}\right)^2\right]\mathrm{d}S$$
$$- \int_0^{L(t)} \frac{EI}{2}\left(\frac{\partial^2 x_2}{\partial S^2}\right)^2\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right]\mathrm{d}S. \qquad (1)$$

Subtituting the relations derived from the inextensibility condition [3]:

$$\partial x_1/\partial S = \sqrt{1 - (\partial x_2/\partial S)^2} \cong 1 - \tfrac{1}{2}(\partial x_2/\partial S)^2, \qquad (2)$$

and

$$\frac{\partial x_1}{\partial t} = -\int_0^S \frac{\partial x_2}{\partial S}\left[1 + \tfrac{1}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right]\frac{\partial^2 x_2}{\partial t\,\partial S}\,\mathrm{d}S, \qquad (3)$$

into equation (1), one obtains

$$\mathscr{L}_0 = \int_0^{L(t)} \tfrac{1}{2}\rho A\left\{\left[-\int_0^S \frac{\partial x_2}{\partial S}\left[1 + \tfrac{1}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right]\frac{\partial^2 x_2}{\partial t\,\partial S}\,\mathrm{d}S + V\left[1 - \tfrac{1}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right]\right]^2\right.$$
$$\left. + \left(\frac{\partial x_2}{\partial t} + V\frac{\partial x_2}{\partial S}\right)^2\right\}\mathrm{d}S - \int_0^{L(t)} \frac{EI}{2}\left(\frac{\partial^2 x_2}{\partial S^2}\right)^2\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right]^2\mathrm{d}S \qquad (4)$$

In order to map equation (4) onto the fixed domain, one needs the following transformation:

$$\mathscr{S} = S/L \qquad (\text{where } 0 \leqslant \mathscr{S} \leqslant 1), \qquad (5)$$

where $L = L(t)$ is the time varying length of the beam outside the channel and S, t are the independent variables. Transformation (5) shows that a fixed point in space S (through which particles pass) does not map to a fixed point in space $\mathscr{S}$, see Figure 1.

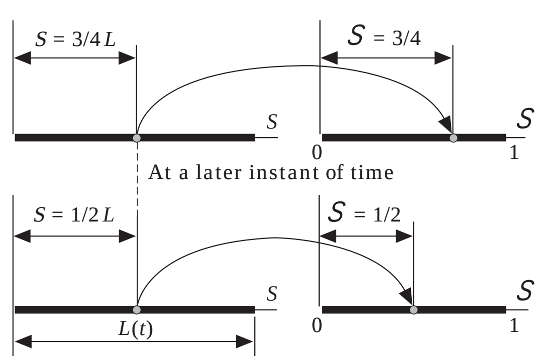

Figure 1. Mapping a fixed point in S to a moving point in $\mathscr{S}$.

From equation (5), one deduces that at time t:

$$dS = L\, d\mathscr{S}, \qquad \frac{\partial x_2}{\partial S} = \frac{\partial \hat{x}_2}{\partial \mathscr{S}} \frac{\partial \mathscr{S}}{\partial S} = \frac{1}{L} \frac{\partial \hat{x}_2}{\partial \mathscr{S}}, \qquad \frac{\partial^2 x_2}{\partial S^2} = \frac{\partial}{\partial \mathscr{S}} \left(\frac{1}{L} \frac{\partial \hat{x}_2}{\partial \mathscr{S}} \right) = \frac{1}{L^2} \frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}, \tag{6}$$

where

$$x_2(S, t) \equiv \hat{x}_2(\mathscr{S}(S, t), t), \tag{7}$$

One notes that the partial operator $\partial(\ )/\partial t$ in the S space provides the time rate of change at a fixed (Eulerian) point S. That is, as various particles pass this point, $\partial(\ )/\partial t|_S$ provides a measure of the changes occurring at this point over the passage of time. The authors have also noted that the fixed Eulerian point S transforms to a *moving* point $\mathscr{S}$ in the $\mathscr{S}$ space. Hence, the $\partial(\ )/\partial t$ operator must be transformed as

$$(\partial/\partial t)(\ )|_S = (\partial/\partial \mathscr{S})(\ )\partial \mathscr{S}/\partial t + (\partial/\partial t)(\ )|_{\mathscr{S}}, \tag{8}$$

Accordingly, one has

$$\partial x_2/\partial t = (\partial \hat{x}_2/\partial \mathscr{S})\, \partial \mathscr{S}/\partial t + \partial \hat{x}_2/\partial t \tag{9}$$

and

$$\frac{\partial^2 x_2}{\partial t^2} = \frac{\partial^2 \mathscr{S}}{\partial t^2} \frac{\partial \hat{x}_2}{\partial \mathscr{S}} + 2 \frac{\partial \mathscr{S}}{\partial t} \frac{\partial^2 \hat{x}_2}{\partial t\, \partial \mathscr{S}} + \left(\frac{\partial \mathscr{S}}{\partial t} \right)^2 \frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} + \frac{\partial^2 \hat{x}_2}{\partial t^2}. \tag{10}$$

From equation (5), it follows that

$$\partial \mathscr{S}/\partial t = -(1/L^2)(\partial L/\partial t) S = -(V/L^2) S = -(V/L)\mathscr{S} \tag{11}$$

and then

$$\begin{aligned} \frac{\partial^2 \mathscr{S}}{\partial t^2} &= (-L(\partial V/\partial t)\mathscr{S} + V^2 \mathscr{S} - V(\partial \mathscr{S}/\partial t)L)/L^2 \\ &= (-L(\partial V/\partial t)\mathscr{S} + V^2 \mathscr{S} - V(-(V/L)\mathscr{S})L)/L^2 \\ &= (2V^2 \mathscr{S} - L(\partial V/\partial t)\mathscr{S})/L^2. \end{aligned} \tag{12}$$

Substituting equations (11) and (12) into equations (9) and (10), one obtains the following two relations:

$$\partial x_2/\partial t = -(V/L)\mathscr{S}\, \partial \hat{x}_2/\partial \mathscr{S} + \partial \hat{x}_2/\partial t \tag{13}$$

and

$$\frac{\partial^2 x_2}{\partial t^2} = \left(2V^2 \mathscr{S} - L \frac{\partial V}{\partial t} \mathscr{S} \right) \Big/ L^2\, (\partial \hat{x}_2/\partial \mathscr{S}) + 2\left(-\frac{V}{L} \mathscr{S} \right) \frac{\partial^2 \hat{x}_2}{\partial t\, \partial \mathscr{S}} + \left(-\frac{V}{L} \mathscr{S} \right)^2 \frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} + \frac{\partial^2 \hat{x}_2}{\partial t^2}. \tag{14}$$

Also using relation (6), one may write

$$\begin{aligned} \frac{\partial^2 x_2}{\partial t\, \partial \mathscr{S}} &= \frac{\partial}{\partial \mathscr{S}} \left(\frac{1}{L} \frac{\partial \hat{x}_2}{\partial \mathscr{S}} \right) \frac{\partial \mathscr{S}}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{L} \frac{\partial \hat{x}_2}{\partial \mathscr{S}} \right) \quad . \\ &= -\frac{V}{L^2} \mathscr{S} \frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} + \frac{1}{L} \frac{\partial^2 \hat{x}_2}{\partial t\, \partial \mathscr{S}} - \frac{V}{L^2} \frac{\partial \hat{x}_2}{\partial \mathscr{S}}. \end{aligned} \tag{15}$$

By substituting relations (13), (14) and (15) into equation (4), the Lagrangian of the sliding flexible beam at instant t, in the fixed domain, can finally be expressed as

$$\begin{aligned}\hat{\mathscr{L}}_0 = \int_0^1 \tfrac{1}{2}\rho A\Bigg\{\Bigg[-\int_0^{\mathscr{S}} \frac{1}{L}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\left[1+\frac{1}{2L^2}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right]\Bigg(-\frac{V}{L^2}\mathscr{S}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}+\frac{1}{L}\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} \\ -\frac{V}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\Bigg)L\,\mathrm{d}\mathscr{S} + V\left(1-\frac{1}{2L^2}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right)\Bigg]^2 \\ +\left(-\frac{V}{L}\mathscr{S}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}+\frac{\partial \hat{x}_2}{\partial t}+\frac{V}{L}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\Bigg\}L\,\mathrm{d}\mathscr{S} \\ -\int_0^1 \frac{EI}{2L^3}\left(\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right)^2\left[1+\frac{1}{L^2}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right]\mathrm{d}\mathscr{S}. \qquad (16)\end{aligned}$$

For the linear case, in the absence of a shortening effect, the Lagrangian (16) may be simplified to the form:

$$\mathscr{L}_{0L} = \int_0^1 \left\{\tfrac{1}{2}\rho A\left[\frac{V}{L}(1-\mathscr{S})\frac{\partial \hat{x}_2}{\partial \mathscr{S}}+\frac{\partial \hat{x}_2}{\partial t}\right]^2 - \frac{EI}{2L^4}\left(\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right)^2\right\}L\,\mathrm{d}\mathscr{S}. \qquad (17)$$

2.2. EQUATION OF MOTION FOR LINEAR AXIALLY INEXTENSIBLE FLEXIBLE SLIDING BEAMS IN THE FIXED DOMAIN

Using a Lagrangian (17) in Hamilton's principle:

$$\delta\int_{t_1}^{t_2} \mathscr{L}_{0L}\,\mathrm{d}t = 0, \qquad (18)$$

and carrying out the variations [3], the equation of motion for the linear axially inextensible flexible sliding beam, in the fixed domain, and in the absence of shortening effect can be expressed as

$$\begin{aligned}\rho A\Bigg\{\frac{\partial^2 \hat{x}_2}{\partial t^2}+\frac{2V}{L}(1-\mathscr{S})\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}}+\frac{V^2}{L^2}(1-\mathscr{S})^2\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} \\ +\left[\frac{2V^2}{L^2}\mathscr{S}-\frac{2V^2}{L^2}+\frac{1}{L}\frac{\partial V}{\partial t}-\frac{1}{L}\frac{\partial V}{\partial t}\mathscr{S}\right]\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\Bigg\}+\frac{EI}{L^4}\frac{\partial^4 \hat{x}_2}{\partial \mathscr{S}^4}=0. \qquad (19)\end{aligned}$$

Now, one needs to account for the shortening effect. The term shortening effect has been used by various researchers with slightly different meanings. It can be seen that as the beam deflects in bending, material points on the beam will have not only a lateral deflection along x_2 but also a secondary motion along the x_1-axis. There will be a kinetic co-energy associated with this secondary motion which must be taken into account. It is to be noted that this secondary motion takes place even when the beam is considered axially inextensible. The additional kinetic co-energy term is considered as

$$\mathscr{L}_{add} = \int_0^{L(t)} \tfrac{1}{2}\rho A\left[-\int_0^S \frac{\partial x_2}{\partial S}\frac{\partial^2 x_2}{\partial t\,\partial S}\,\mathrm{d}S + V\right]^2 \mathrm{d}S. \qquad (20)$$

Here, one may use the Leibniz rule to integrate equation (20). After setting aside the boundary terms and noting the prescribed terms, the additional Lagrangian term that contributes to the equation of motion may be expressed as

$$\mathscr{L}_{add} = \int_0^{L(t)} \tfrac{1}{2}\rho A \frac{\partial V}{\partial t}(L - S)\left(\frac{\partial x_2}{\partial S}\right)^2 \mathrm{d}S. \tag{21}$$

Using relations (6), equation (21) may also be transformed to the fixed domain and is given by

$$\hat{\mathscr{L}}_{add} = \int_0^{1} \tfrac{1}{2}\rho A \frac{\partial V}{\partial t}(1 - S)\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2 \mathrm{d}S. \tag{22}$$

Now adding $\hat{\mathscr{L}}_{add}$ to $\hat{\mathscr{L}}_{0L}$ and carrying out the variation, the comprehensive form of equation of motion of linear sliding flexible beam, including the shortening effect, in the fixed domain is obtained as

$$\rho A\left\{\frac{\partial^2 \hat{x}_2}{\partial t^2} + \frac{2V}{L}(1-\mathscr{S})\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} + \left[\frac{V^2}{L^2}(1-\mathscr{S})^2 + \frac{1}{L}\frac{\partial V}{\partial t}(1-\mathscr{S})\right]\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right.$$
$$\left. + \left[\frac{2V^2}{L^2}\mathscr{S} - \frac{2V^2}{L^2} - \frac{1}{L}\frac{\partial V}{\partial t}\mathscr{S}\right]\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right\} + \frac{EI}{L^4}\frac{\partial^4 \hat{x}_2}{\partial \mathscr{S}^4} = 0. \tag{23}$$

The solution of the above equation and its geometric and natural boundary conditions, as derived in part I, lead to the transient response of the linear sliding flexible beams.

2.3. SOLUTION PROCEDURE: GALERKIN'S METHOD

One way of solving equation (23), is to use Galerkin's method. To this end one first transforms the governing equation of motion into a non-dimensional form. Introducing the non-dimensional quantities:

$$\eta = \hat{x}_2/L_0, \qquad \tau = (EI/\rho A)^{1/2} t/L_0^2, \qquad v = (\rho A/EI)^{1/2} V L_0, \tag{24}$$

where

$$L(t) = L_0 l(t),$$

and using the relations (24) in equation (23), after some manipulations, the dimension-less governing equation of motion is obtained as

$$\frac{\partial^2 \eta}{\partial \tau^2} + \frac{2v(1-\mathscr{S})}{l}\frac{\partial^2 \eta}{\partial \tau\,\partial \mathscr{S}} + \left[\frac{v^2(1-\mathscr{S})^2}{l^2} + \frac{1}{l}\frac{\partial v}{\partial \tau}(1-\mathscr{S})\right]\frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$
$$-\frac{1}{l^2}\left[l\frac{\partial v}{\partial \tau}\mathscr{S} + (2 - 2\mathscr{S})v^2\right]\frac{\partial \eta}{\partial \mathscr{S}} + \frac{1}{l^4}\frac{\partial^4 \eta}{\partial \mathscr{S}^4} = 0. \tag{25}$$

To obtain a solution, one expresses η in terms of modal functions $\mathscr{P}_j(\mathscr{S})$ as

$$\eta(\mathscr{S}, \tau) = \sum_{j=1}^{N} \mathscr{P}_j(\mathscr{S}) q_j(\tau), \tag{26}$$

where $q_j(\tau)$ are the unknown modal coefficients. Substituting from equation (26) into equation (25) and using Galerkin's method, a set of ODEs in the time part of the solution $q_j(\tau)$ is obtained. For the moment is must be implicitly assumed that the solution η converges to the true solution as $N \to \infty$. On physical grounds one can anticipate that for problems with the majority of energy at low frequencies, the motion in low modes will predominate and that such an approximation will thus converge rapidly onto the true solution. Substituting relation (26) into equation (25) and following Galerkin's method, i.e., multiplying the residual obtained by $\mathscr{P}_i(\mathscr{S})$ and integrating over the beam length (0, 1), one may obtain the second order ODE as

$$\ddot{q}_i + a_{ij}\dot{q}_j + b_{ij}q_j = 0, \tag{27}$$

where

$$a_{ij} = \int_0^1 \frac{2v}{l}(1-\mathscr{S})\mathscr{P}_i \frac{\partial \mathscr{P}_j}{\partial \mathscr{S}} \mathrm{d}\mathscr{S} \tag{28}$$

and

$$b_{ij} = \int_0^1 \mathscr{P}_i \left\{ \left[\frac{v^2}{l^2}(1-\mathscr{S})^2 + \frac{1}{l}\frac{\partial v}{\partial \tau}(1-\mathscr{S}) \right] \frac{\partial^2 \mathscr{P}_j}{\partial \mathscr{S}^2} - \frac{1}{l^2}\left[l \frac{\partial v}{\partial \tau}\mathscr{S} + 2(1-\mathscr{S})v^2 \right] \frac{\partial \mathscr{P}_j}{\partial \mathscr{S}} + \frac{1}{l^4}\frac{\partial^4 \mathscr{P}_j}{\partial \mathscr{S}^4} \right\} \mathrm{d}\mathscr{S}. \tag{29}$$

The coefficients a_{ij} and b_{ij} are time dependent making the governing equations of motion non-autonomous. It is important to note the significant advantage of deriving these equations in the fixed domain since for the computation of a_{ij} and b_{ij} the space dependent eigenfunctions of the beam instead of time and space dependent eigenfunctions, as used by other researchers [4], [5] may be used.

2.4. TIME INTEGRATION OF THE SYSTEM EQUATIONS

To integrate the system equations (27) in time and obtain the transient response of the system, it is required to define a proper set of comparison functions [6]. For an axially rigid flexible sliding beam emerging from a fixed rigid channel, the ortho-normal eigenfunctions of the stationary cantilever provide a convenient set of comparison functions in the form:

$$\mathscr{P}_i = \cos\beta_i\eta - \cosh\beta_i\eta - \frac{\cos\beta_i + \cosh\beta_i}{\sin\beta_i + \sinh\beta_i}(\sin\beta_i\eta - \sinh\beta_i\eta), \tag{30}$$

where the eigenvalues β_i are roots of the characteristic equation

$$1 + \cos\beta_i \cosh\beta_i = 0. \tag{31}$$

Since in the fixed domain the the comparison functions are only space dependent, by substituting equation (30) into relations (28) and (29) and using the MAPLE symbolic program, one can integrate and determine the matrices a_{ij} and b_{ij} as functions of prescribed velocity of the sliding beam and the instantaneous length of the beam.

With the matrices a_{ij} and b_{ij} at hand, one may now integrate the set of ODEs in equation (27) and obtain the transient response of the system. This has been done using the IMSL/DIVPRK routine.

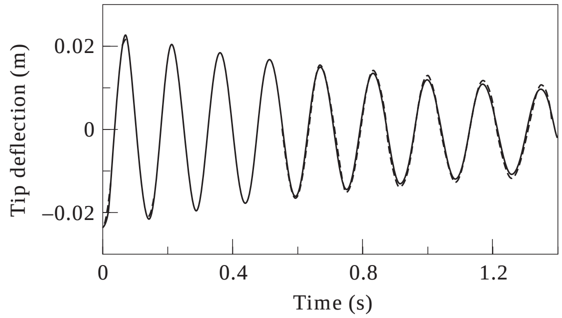

Figure 2. Reverse spaghetti problem: constant velocity extrusion; ——, Behdinan; ---, Stylianou.

To validate the equations and their solutions, some examples studied by Stylianou [7] are considered. These cases were also investigated experimentally and simulated by Yuh and Young [8]. To compare results one need to define a non-negative, structural damping force proportional to $\dot{q}_j$. Such a damping coefficient was determined experimentally by Yuh and Young. In this case of variable length beam, the damping coefficient becomes length dependent. Stylianou calculated an equivalent Rayleigh damping coefficient for his discretized equations and was able to compare his results with the experimental responses determined by Yuh and Young. For the present formulation of the sliding beam, *in the fixed domain*, it is not evident how the damping coefficient (which is inserted in the discretized equations and does not appear in the partial differential equation) should be calculated. An indirect and relatively easy approach is to "guess" a damping coefficient and on comparison of results with those of Stylianou, adjust the coefficient for excellent agreement for small amplitude oscillations. Once this is accomplished, the same coefficient may be used for large amplitudes of oscillations.

2.4.1. *Spaghetti and reverse spaghetti problems*: *quadratic sliding motion*

Let one consider a quadratic sliding motion as

$$L(t) = L_0 + V_0 t + a_0 t^2/2. \tag{32}$$

In the case of the reverse spaghetti problem, i.e., a beam extruding from a channel, a constant velocity extrusion with the initial length $L_0 = 0{\cdot}4255$ m and velocity $V_0 = 0{\cdot}0410$ m/s is examined. The beam properties are: $\rho = 3144{\cdot}3858$ kg/m^3, $E = 68{\cdot}96 \times 10^9$ N/m^2, $A = 4{\cdot}3434 \times 10^{-5}$ m^2 and $I = 1{\cdot}059 \times 10^{-11}$ m^4. The time history of the tip deflection is given in Figure 2, which is in good agreement with the results obtained in test case 2 by Stylianou [7].

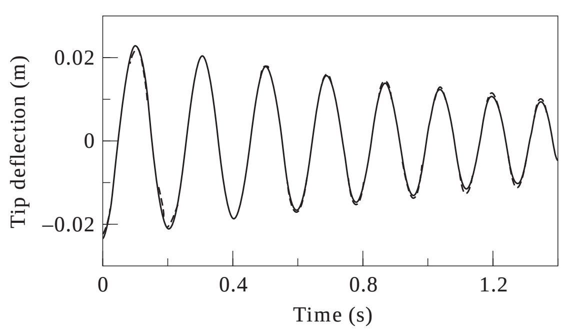

Figure 3. Spaghetti problem: constant acceleration retraction; ——, Behdinan; ---, Stylianou.

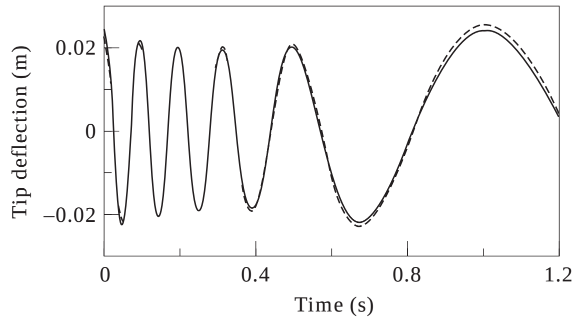

Figure 4. Reverse spaghetti problem: low frequency extrusion; ——, Behdinan; ---, Stylianou.

For the spaghetti problem, i.e., a beam retracting into a channel, one considers a constant acceleration retracting beam with the initial length $L_0 = 0{\cdot}521$ m and velocity $V_0 = -0{\cdot}0300$ m/s and constant acceleration $a_0 = -0{\cdot}0540$ m/s^2. Figure 3 shows the tip response of the beam which is in good agreement with the results obtained in test case 3 of Stylianou [7].

2.4.2. *Spaghetti and reverse spaghetti problems*: *repositional motion*

For these simulations, the beam's length is described by

$$L(t) = L_0 + c_0/t_0[t - (t_0/2\pi)\sin(2\pi t/t_0)], \tag{33}$$

where in the case of the extrusion $L_0 = 0{\cdot}35$ m, $c_0 = 0{\cdot}7$ m and $t_0 = 1{\cdot}2$ s and for retraction these parameters are given as: $L_0 = 1{\cdot}05$ m, $c_0 = -0{\cdot}7$ m and $t_0 = 1{\cdot}2$ s. Equation (33) allows us to consider axial motions made up of a constant and an oscillatory part. For the oscillatory part it is of interest to examine frequencies higher and lower than the fundamental natural frequency of the beam at the initial configuration, for the retraction case, and the final configuration, for the extrusion case. The results are given in Figures 4 and 5 which are again in good agreement with those obtained in simulations 1 and 2 by Stylianou [7].

In the case of high frequency oscillation where the period of oscillation $t_0 = 0{\cdot}2$ s, with the same parameters, the results shown in Figures 6 and 7 are in good qualitative agreement with those obtained in simulation cases 5 and 7 by Yuh and Young [8].

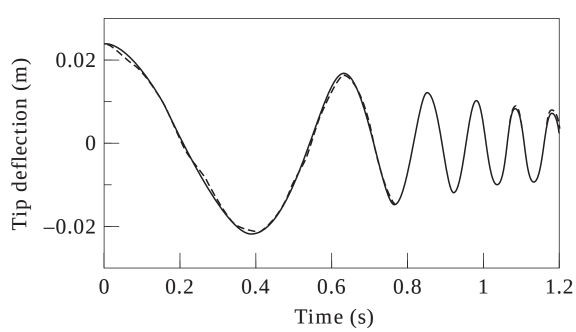

Figure 5. Spaghetti problem: low frequency retraction; ——, Behdinan; ---, Stylianou.

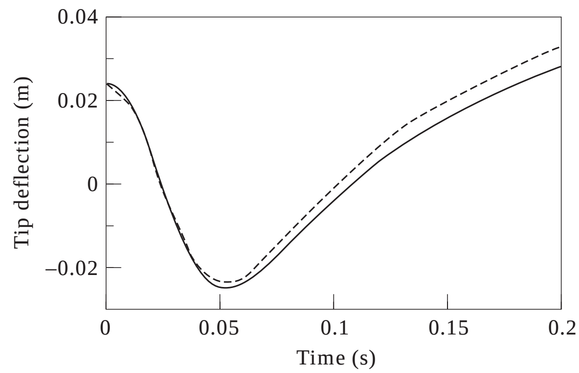

Figure 6. Reverse spaghetti problem: high frequency extrusion; ——, Behdinan; ---, Stylianou.

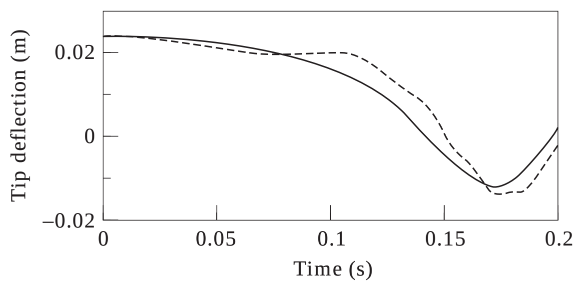

Figure 7. Spaghetti problem: high frequency retraction; ——, Behdinan; ---, Stylianou.

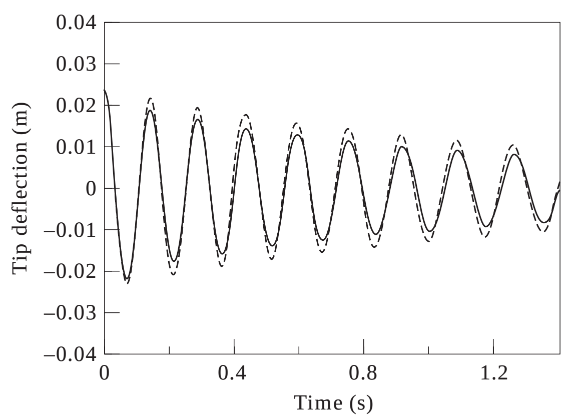

Figure 8. Reverse spaghetti problem: high frequency perturbation; ——, perturbed; ---, unperturbed.

2.4.3. *Reverse spaghetti problem*: *high frequency perturbation*

It is possible to attenuate the transverse oscillations of flexible sliding beams by introducing a high frequency, low amplitude perturbation to the otherwise constant-velocity axial motion. This was addressed by Golnaraghi [9] and Stylianou [7].

In this case the time varying length of the beam can be expressed as

$$L(t) = L_0(1 + \varepsilon \sin \hat{\omega} t) + V_0 t. \tag{34}$$

Zajaczkowski and Lipinski [10] used Bolotin's approach [11] to obtain the region of stability and instability for this problem. Based on the perturbation parameters ε and $\sqrt{\omega_1/2\hat{\omega}}$, where ω_1 is the instantaneous fundamental natural frequency of the sliding beam, we may have a stable or unstable response. Here one considers the following parameters in the stable region: $L_0 = 0{\cdot}4255$ m, $\varepsilon = 0{\cdot}0133$ and $\hat{\omega} = 11\omega_1$. Figure 8 shows the resulting suppression of oscillation. These simulations clearly demonstrate the change in frequency and more importantly the amplitude of the oscillations due to the changing length of the beam.

3. NON-LINEAR AXIALLY INEXTENSIBLE FLEXIBLE SLIDING BEAMS IN THE FIXED DOMAIN

To study the transient response of the non-linear axially inextensible beam, the governing equation of motion is mapped to the fixed domain and then using Galerkin's approach the system equations are discretized and obtain a set of ODEs in $q_i(\tau)$ coefficients. Returning to equation (26), one then obtains the response of the system in the fixed domain. Subsequently one may transform the response to the physical i.e., variable domain.

3.1. NON-LINEAR EQUATION OF MOTION OF INEXTENSIBLE FLEXIBLE SLIDING BEAMS IN THE FIXED DOMAIN

Here as in section 2, one needs to use a one to one mapping from the time variable domain to the fixed domain. Employing the transformation defined in equation (5) and using the relations (6), (13–15) in the equation of motion of the axially inextensible sliding beam given in part I, one obtains

$$L_1 + N_1 + N_2 = 0, \tag{35}$$

where

$$L_1 = \rho A\left\{\frac{\partial^2 \hat{x}_2}{\partial t^2} + \frac{2V}{L}(1-\mathscr{S})\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} + \left[\frac{V^2}{L^2}(1-\mathscr{S})^2 + \frac{1}{L}\frac{\partial V}{\partial t}(1-\mathscr{S})\right]\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right.$$
$$\left. + \left[\frac{2V^2}{L^2}\mathscr{S} - \frac{2V^2}{L^2} - \frac{1}{L}\frac{\partial V}{\partial t}\mathscr{S}\right]\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right\} + \frac{EI}{L^4}\frac{\partial^4 \hat{x}_2}{\partial \mathscr{S}^4}, \tag{36}$$

$$N_1 = \rho A\left\{\frac{2V}{L^3}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} + \left[\frac{-2V^2\mathscr{S}}{L^4} + \frac{3}{2L^3}\frac{\partial V}{\partial t}(1-\mathscr{S}) + \frac{V^2}{L^4}\right]\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right.$$
$$\left. - \frac{2V^2}{L^4}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^3\right\} + \frac{EI}{L^6}\left[\frac{\partial^4 \hat{x}_2}{\partial \mathscr{S}^4}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2 + 4\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\frac{\partial^3 \hat{x}_2}{\partial \mathscr{S}^3} + \left(\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right)^3\right], \tag{37}$$

$$N_2 = \frac{1}{L}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\int_0^{\mathscr{S}} \rho A\left[\left(\frac{-V\mathscr{S}}{L^2}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} + \frac{1}{L}\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} - \frac{V}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right.$$
$$+ \frac{1}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\left(\frac{2V^2 - L(\partial V/\partial t)}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}} + \frac{4V^2\mathscr{S} - L(\partial V/\partial t)\mathscr{S}}{L^2}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} - \frac{2V}{L}\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}}\right.$$

$$- \frac{2V\mathscr{S}}{L}\frac{\partial^3 \hat{x}_2}{\partial t\,\partial \mathscr{S}^2} + \frac{V^2\mathscr{S}^2}{L^2}\frac{\partial^3 \hat{x}_2}{\partial \mathscr{S}^3} + \frac{\partial^3 \hat{x}_2}{\partial t^2\,\partial \mathscr{S}}\bigg)\bigg]L\,\mathrm{d}\mathscr{S}$$

$$- \frac{1}{L^2}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\left\{\int_{\mathscr{S}}^1 \int_0^{\mathscr{S}} \rho A\left[\left(\frac{-V\mathscr{S}}{L^2}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} + \frac{1}{L}\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} - \frac{V}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right.\right.$$

$$+ \frac{1}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\left(\frac{2V^2 - L(\partial V/\partial t)}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}} + \frac{4V^2\mathscr{S} - L(\partial V/\partial t)\mathscr{S}}{L^2}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right.$$

$$- \frac{2V}{L}\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}} - \frac{2V\mathscr{S}}{L}\frac{\partial^3 \hat{x}_2}{\partial t\,\partial \mathscr{S}^2} + \frac{V^2\mathscr{S}^2}{L^2}\frac{\partial^3 \hat{x}_2}{\partial \mathscr{S}^3} + \frac{\partial^3 \hat{x}_2}{\partial t^2\,\partial \mathscr{S}}\bigg)\bigg]L^2\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S}$$

$$+ \int_{\mathscr{S}}^1 \rho A\left[\frac{1}{2L^2}\frac{\partial V}{\partial t}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2 + \frac{2V}{L}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\left(\frac{-V\mathscr{S}}{L^2}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} + \frac{1}{L}\frac{\partial^2 \hat{x}_2}{\partial t\,\partial \mathscr{S}}\right.\right.$$

$$\left.\left.\left. - \frac{V}{L^2}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right) + \frac{V^2}{L^3}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}\right]L\,\mathrm{d}\mathscr{S}\right\}. \tag{38}$$

In equation (35), L_1, N_1 and N_2 contain linear, non-linear and non-linear integral terms, respectively.

Introducing the non-dimensional quantities in equation (24), we may express the non-dimensional form of the governing equation of motion for the inextensible sliding beam, in the fixed domain, as

$$\mathscr{L}(\eta) + \mathscr{N}_1(\eta) + \mathscr{N}_2(\eta) = 0, \tag{39}$$

where

$$\mathscr{L}(\eta) = \frac{\partial^2 \eta}{\partial \tau^2} + \frac{2v}{l}(1-\mathscr{S})\frac{\partial^2 \eta}{\partial \tau\,\partial \mathscr{S}} + \left[\frac{v^2}{l^2}(1-\mathscr{S})^2 + \frac{1}{l}\frac{\partial v}{\partial \tau}(1-\mathscr{S})\right]\frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$+ \left[\frac{2v^2}{l^2}\mathscr{S} - \frac{2v^2}{l^2} - \frac{1}{l}\frac{\partial v}{\partial \tau}\mathscr{S}\right]\frac{\partial \eta}{\partial \mathscr{S}}\bigg\} + \frac{1}{l^4}\frac{\partial^4 \eta}{\partial \mathscr{S}^4}, \tag{40}$$

$$\mathscr{N}_1(\eta) = \frac{2v}{l^3}\left(\frac{\partial \eta}{\partial \mathscr{S}}\right)^2\frac{\partial^2 \eta}{\partial \tau\,\partial \mathscr{S}} + \left[\frac{-2v^2\mathscr{S}}{l^4} + \frac{3}{2l^3}\frac{\partial v}{\partial \tau}(1-\mathscr{S}) + \frac{v^2}{l^4}\right]\left(\frac{\partial \eta}{\partial \mathscr{S}}\right)^2\frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$- \frac{2v^2}{l^4}\left(\frac{\partial \eta}{\partial \mathscr{S}}\right)^3 + \frac{1}{l^6}\left[\frac{\partial^4 \eta}{\partial \mathscr{S}^4}\left(\frac{\partial \eta}{\partial \mathscr{S}}\right)^2 + 4\frac{\partial \eta}{\partial \mathscr{S}}\frac{\partial^2 \eta}{\partial \mathscr{S}^2}\frac{\partial^3 \eta}{\partial \mathscr{S}^3} + \left(\frac{\partial^2 \eta}{\partial \mathscr{S}^2}\right)^3\right] \tag{41}$$

and

$$\mathscr{N}_2 = \frac{\partial \eta}{\partial \mathscr{S}}\int_0^{\mathscr{S}}\left(\frac{-v\mathscr{S}}{l^2}\frac{\partial^2 \eta}{\partial \mathscr{S}^2} + \frac{1}{l}\frac{\partial^2 \eta}{\partial \tau\,\partial \mathscr{S}} - \frac{v}{l^2}\frac{\partial \eta}{\partial \mathscr{S}}\right)^2$$

$$+ \frac{1}{l^2}\frac{\partial \eta}{\partial \mathscr{S}}\left(\frac{2v^2 - l(\partial v/\partial \tau)}{l^2}\frac{\partial \eta}{\partial \mathscr{S}} + \frac{4v^2\mathscr{S} - l(\partial v/\partial \tau)\mathscr{S}}{l^2}\frac{\partial^2 \eta}{\partial \mathscr{S}^2} - \frac{2v}{l}\frac{\partial^2 \eta}{\partial \tau\,\partial \mathscr{S}}\right.$$

$$- \frac{2v\mathscr{S}}{l}\frac{\partial^3\eta}{\partial\tau\,\partial\mathscr{S}^2} + \frac{v^2\mathscr{S}^2}{l^2}\frac{\partial^3\eta}{\partial\mathscr{S}^3} + \frac{\partial^3\eta}{\partial\tau^2\,\partial\mathscr{S}}\bigg)\bigg]\,\mathrm{d}\mathscr{S}$$

$$- \frac{\partial^2\eta}{\partial\mathscr{S}^2}\left\{\int_{\mathscr{S}}^1\int_0^{\mathscr{S}}\left[\left(\frac{-v\mathscr{S}}{l^2}\frac{\partial^2\eta}{\partial\mathscr{S}^2} + \frac{1}{l}\frac{\partial^2\eta}{\partial\tau\,\partial\mathscr{S}} - \frac{v}{l^2}\frac{\partial\eta}{\partial\mathscr{S}}\right)^2\right.\right.$$

$$+ \frac{1}{l^2}\frac{\partial\eta}{\partial\mathscr{S}}\left(\frac{2v^2 - l(\partial v/\partial\tau)}{l^2}\frac{\partial\eta}{\partial\mathscr{S}} + \frac{4v^2\mathscr{S} - l(\partial v/\partial\tau)\mathscr{S}}{l^2}\frac{\partial^2\eta}{\partial\mathscr{S}^2}\right.$$

$$\left.\left. - \frac{2v}{l}\frac{\partial^2\eta}{\partial\tau\,\partial\mathscr{S}} - \frac{2v\mathscr{S}}{l}\frac{\partial^3\eta}{\partial\tau\,\partial\mathscr{S}^2} + \frac{v^2\mathscr{S}^2}{l^2}\frac{\partial^3\eta}{\partial\mathscr{S}^3} + \frac{\partial^3\eta}{\partial\tau^2\,\partial\mathscr{S}}\right)\right]\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S}$$

$$+ \int_{\mathscr{S}}^1\left[\frac{1}{2l^3}\frac{\partial v}{\partial\tau}\left(\frac{\partial\eta}{\partial\mathscr{S}}\right)^2 + \frac{2v}{l^2}\frac{\partial\eta}{\partial\mathscr{S}}\left(\frac{-v\mathscr{S}}{l^2}\frac{\partial^2\eta}{\partial\mathscr{S}^2} + \frac{1}{l}\frac{\partial^2\eta}{\partial\tau\,\partial\mathscr{S}}\right.\right.$$

$$\left.\left.\left. - \frac{v}{l^2}\frac{\partial\eta}{\partial\mathscr{S}}\right) + \frac{v^2}{l^4}\frac{\partial\eta}{\partial\mathscr{S}}\frac{\partial^2\eta}{\partial\mathscr{S}^2}\right]\mathrm{d}\mathscr{S}\right\}. \tag{42}$$

The non-linear terms

$$\int_0^{\mathscr{S}}\frac{\partial\eta}{\partial\mathscr{S}}\frac{\partial^3\eta}{\partial\tau^2\,\partial\mathscr{S}}\,\mathrm{d}\mathscr{S} \qquad \text{and} \qquad \int_{\mathscr{S}}^1\int_0^{\mathscr{S}}\frac{\partial\eta}{\partial\mathscr{S}}\frac{\partial^3\eta}{\partial\tau^2\,\partial\mathscr{S}}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S}$$

create some difficulties in the conventional solution procedures for dynamical systems. In most cases the equations of motion have linear second order time derivative inertia terms. One approach for dealing with these terms has been proposed by Li and Païdoussis [12]. In this approach one approximates these troublesome terms by first noting that while the beam deflection can be large, from a practical point of view, only the values of deflection much smaller than the length of the beam, i.e., $\eta \ll 1$, need be considered. To identify the relative magnitudes of various terms, one replaces η by $\sqrt{\varepsilon}\,\eta$, where $\varepsilon \ll 1$, in equation (39). This leads to

$$\mathscr{L}(\eta) + \varepsilon\mathscr{N}_1(\eta) + \varepsilon\mathscr{N}_2(\eta) = 0. \tag{43}$$

It is now evident that the non-linear terms are considerably smaller than the linear terms. Thus as a first approximation one may solve

$$\mathscr{L}(\eta) = 0 \tag{44}$$

and from this expression the term $\partial^2\eta/\partial\tau^2$ may be isolated in equation (44) and expressions obtained for

$$\int_0^{\mathscr{S}}\frac{\partial\eta}{\partial\mathscr{S}}\frac{\partial^3\eta}{\partial\tau^2\,\partial\mathscr{S}}\,\mathrm{d}\mathscr{S} \qquad \text{and} \qquad \int_{\mathscr{S}}^1\int_0^{\mathscr{S}}\frac{\partial\eta}{\partial\mathscr{S}}\frac{\partial^3\eta}{\partial\tau^2\,\partial\mathscr{S}}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S}$$

as:

$$\int_0^{\mathscr{S}} \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \tau^2 \, \partial \mathscr{S}} \, \mathrm{d}\mathscr{S} = -\int_0^{\mathscr{S}} \left\{ \frac{2v}{l} (1-\mathscr{S}) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \tau \, \partial \mathscr{S}^2} - \frac{2v}{l} \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} \right.$$

$$- \left[\frac{1}{l} \frac{\partial v}{\partial \tau} (1+\mathscr{S}) + \frac{4v^2}{l^2} (1-\mathscr{S}) \right] \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$+ \left[\frac{1}{l} \frac{\partial v}{\partial \tau} + \frac{v^2}{l^2} (1-\mathscr{S}) \right] (1-\mathscr{S}) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \mathscr{S}^3}$$

$$\left. - \frac{1}{l^2} \left(l \frac{\partial v}{\partial \tau} - 2v^2 \right) \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^2 + \frac{1}{l^4} \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^5 \eta}{\partial \mathscr{S}^5} \right\} \mathrm{d}\mathscr{S}. \tag{45}$$

The other non-linear term may be obtained by integrating equation (45) from $\mathscr{S}$ to 1.

Substituting the transformed non-linear inertial term into equation (43) and returning to the original variable η, after very laborious manipulations, one obtains the non-linear partial integro-differential equation of motion of the inextensible flexible sliding beam as

$$\frac{\partial^2 \eta}{\partial \tau^2} + \frac{2v}{l} (1-\mathscr{S}) \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} + \left[\frac{v^2}{l^2} (1-\mathscr{S})^2 + \frac{1}{l} \frac{\partial v}{\partial \tau} (1-\mathscr{S}) \right] \frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$+ \left[\frac{2v^2}{l^2} \mathscr{S} - \frac{2v^2}{l^2} - \frac{1}{l} \frac{\partial v}{\partial \tau} \mathscr{S} \right] \frac{\partial \eta}{\partial \mathscr{S}} + \frac{1}{l^4} \frac{\partial^4 \eta}{\partial \mathscr{S}^4} + \mathscr{N}(\eta) = 0, \tag{46}$$

where

$$\mathscr{N}(\eta) = \mathscr{N}^1(\eta) + \mathscr{N}^2(\eta) \tag{47}$$

and

$$\mathscr{N}^1(\eta) = \frac{2v}{l^3} \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^2 \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} + \left[\frac{v^2}{l^4} (1-2\mathscr{S}) + \frac{3}{2l^3} \frac{\partial v}{\partial \tau} (1-\mathscr{S}) \right] \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^2 \frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$- \frac{2v^2}{l^4} \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^3 + \frac{1}{l^6} \left[3 \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} \frac{\partial^3 \eta}{\partial \mathscr{S}^3} + \left(\frac{\partial^2 \eta}{\partial \mathscr{S}^2} \right)^3 \right]$$

$$+ \frac{\partial \eta}{\partial \mathscr{S}} \int_0^{\mathscr{S}} \left\{ \left(\frac{-v\mathscr{S}}{l^2} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} + \frac{1}{l} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} - \frac{v}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} \right)^2 \right.$$

$$+ \frac{1}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} \left(\frac{2v^2 - l(\partial v/\partial \tau)}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} + \frac{4v^2\mathscr{S} - l(\partial v/\partial \tau)\mathscr{S}}{l^2} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} \right.$$

$$\left. - \frac{2v}{l} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} - \frac{2v\mathscr{S}}{l} \frac{\partial^3 \eta}{\partial \tau \, \partial \mathscr{S}^2} + \frac{v^2 \mathscr{S}^2}{l^2} \frac{\partial^3 \eta}{\partial \mathscr{S}^3} \right)$$

$$- \frac{1}{l^2} \left[\frac{2v}{l} (1-\mathscr{S}) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \tau \, \partial \mathscr{S}^2} - \frac{2v}{l} \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} \right.$$

$$- \left(\frac{1}{l} \frac{\partial v}{\partial \tau} (1 + \mathscr{S}) + \frac{4v^2}{l^2} (1 - \mathscr{S}) \right) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$+ \left[\frac{1}{l} \frac{\partial v}{\partial \tau} + \frac{v^2}{l^2} (1 - \mathscr{S}) \right] (1 - \mathscr{S}) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \mathscr{S}^3}$$

$$- \frac{1}{l^2} \left(l \frac{\partial v}{\partial \tau} - 2v^2 \right) \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^2 - \frac{1}{l^4} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} \frac{\partial^4 \eta}{\partial \mathscr{S}^4} \Bigg] \Bigg\} \, \mathrm{d}\mathscr{S}, \tag{48}$$

$$\mathscr{N}^2(\eta) = -\frac{\partial^2 \eta}{\partial \mathscr{S}^2} \Bigg\{ \int_{\mathscr{S}}^{1} \int_{0}^{\mathscr{S}} \Bigg[\frac{-v\mathscr{S}}{l^2} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} + \frac{1}{l} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} - \frac{v}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} \Bigg)^2 + \frac{1}{l^2} \frac{\partial \eta}{\partial \mathscr{S}}$$

$$\times \Bigg(\frac{2v^2 - l(\partial v / \partial \tau)}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} + \frac{4v^2 \mathscr{S} - l(\partial v / \partial t)\mathscr{S}}{l^2} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} - \frac{2v}{l} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}}$$

$$- \frac{2v\mathscr{S}}{l} \frac{\partial^3 \eta}{\partial \tau \, \partial \mathscr{S}^2} + \frac{v^2 \mathscr{S}^2}{l^2} \frac{\partial^3 \eta}{\partial \mathscr{S}^3} \Bigg) - \frac{1}{l^2} \Bigg(\frac{2v}{l} (1 - \mathscr{S}) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \tau \, \partial \mathscr{S}^2}$$

$$- \frac{2v}{l} \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} - \left(\frac{1}{l} \frac{\partial v}{\partial \tau} (1 + \mathscr{S}) + \frac{4v^2}{l^2} (1 - \mathscr{S}) \right) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$+ \left(\frac{1}{l} \frac{\partial v}{\partial \tau} + \frac{v^2}{l^2} (1 - \mathscr{S}) \right) (1 - \mathscr{S}) \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^3 \eta}{\partial \mathscr{S}^3} - \frac{1}{l^2} \left(l \frac{\partial v}{\partial \tau} - 2v^2 \right) \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^2$$

$$- \frac{1}{l^4} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} \frac{\partial^4 \eta}{\partial \mathscr{S}^4} \Bigg) \Bigg] \, \mathrm{d}\mathscr{S} \, \mathrm{d}\mathscr{S} + \int_{\mathscr{S}}^{1} \Bigg[\frac{1}{2l^3} \frac{\partial v}{\partial \tau} \left(\frac{\partial \eta}{\partial \mathscr{S}} \right)^2 + \frac{2v}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} \Bigg(\frac{-v\mathscr{S}}{l^2} \frac{\partial^2 \eta}{\partial \mathscr{S}^2}$$

$$+ \frac{1}{l} \frac{\partial^2 \eta}{\partial \tau \, \partial \mathscr{S}} - \frac{v}{l^2} \frac{\partial \eta}{\partial \mathscr{S}} \Bigg) + \frac{v^2}{l^4} \frac{\partial \eta}{\partial \mathscr{S}} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} + \frac{1}{l^6} \frac{\partial^2 \eta}{\partial \mathscr{S}^2} \frac{\partial^3 \eta}{\partial \mathscr{S}^3} \Bigg) \, \mathrm{d}S \Bigg\}. \tag{49}$$

To obtain the transient response of the inextensible flexible sliding beams, one needs to solve the above integro-differential equation in the fixed domain and transform the response back to the time variable domain.

3.2. TRANSFORMATION OF THE PDE TO A SET OF ODES: GALERKIN'S METHOD

With reference to the discussion in section 2, one may use Galerkin's method to solve the governing PDE. Substituting for η from equation (26) in terms of modal functions $\mathscr{P}_j(\mathscr{S})$ and the corresponding generalized coordinates $q_j(\tau)$ into the governing equation of motion (46), multiplying by $\mathscr{P}_i(\mathscr{S})$ and integrating from 0 to 1, one obtains the desired ODEs as

$$\ddot{q}_i + a_{ij}\dot{q}_j + b_{ij}q_j + c_{ijkl}q_jq_kq_l + d_{ijkl}q_jq_k\dot{q}_l + e_{ijkl}q_j\dot{q}_k\dot{q}_l = 0, \tag{50}$$

where

$$a_{ij} = \int_0^1 \frac{2v}{l}(1-\mathscr{S})\mathscr{P}_i\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\,\mathrm{d}\mathscr{S}, \tag{51}$$

$$b_{ij} = \int_0^1 \mathscr{P}_i\left\{\left[\frac{v^2}{l^2}(1-\mathscr{S})^2 + \frac{1}{l}\frac{\partial v}{\partial \tau}(1-\mathscr{S})\right]\frac{\partial^2\mathscr{P}_j}{\partial \mathscr{S}^2}\right.$$
$$\left. -\frac{1}{l^2}\left[l\frac{\partial v}{\partial \tau}\mathscr{S} + 2(1-\mathscr{S})v^2\right]\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}} + \frac{1}{l^4}\frac{\partial^4\mathscr{P}_j}{\partial \mathscr{S}^4}\right\}\mathrm{d}\mathscr{S}, \tag{52}$$

$$c_{ijkl} = \int_0^1 \mathscr{P}_i\left\{\left[\frac{v^2}{l^4}(1-2\mathscr{S}) + \frac{3}{2l^3}\frac{\partial v}{\partial \tau}(1-\mathscr{S})\right]\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2} - \frac{2v^2}{l^4}\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\right.$$
$$+\frac{1}{l^6}\left(3\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^3 \mathscr{P}_l}{\partial \mathscr{S}^3} + \frac{\partial^2 \mathscr{P}_j}{\partial \mathscr{S}^2}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2}\right) + \frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\int_0^{\mathscr{S}}\left[\left(\frac{v^2\mathscr{S}^2}{l^4}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2}\right.\right.$$
$$\left.+\frac{v^2}{l^4}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}} + \frac{2v^2\mathscr{S}}{l^4}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\right) + \frac{1}{l^2}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\left(\frac{1}{l^2}\left(l\frac{\partial v}{\partial \tau} + 4v^2\right)\right.$$
$$\left.\times\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2} + \left(\frac{v^2\mathscr{S}^2}{l^2} - \frac{1}{l}\frac{\partial v}{\partial \tau}(1-\mathscr{S}) - \frac{v^2(1-\mathscr{S})^2}{l^2}\right)\frac{\partial^3 \mathscr{P}_l}{\partial \mathscr{S}^3}\right)$$
$$\left.+\frac{1}{l^6}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^4 \mathscr{P}_l}{\partial \mathscr{S}^4}\right]\mathrm{d}\mathscr{S} - \frac{\partial^2 \mathscr{P}_j}{\partial \mathscr{S}^2}\left[\int_{\mathscr{S}}^1\int_0^{\mathscr{S}}\left(\left(\frac{v^2\mathscr{S}^2}{l^4}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2}\right.\right.\right.$$
$$\left.+\frac{v^2}{l^4}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}} + \frac{2v^2\mathscr{S}}{l^4}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\right) + \frac{1}{l^2}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\left(\frac{1}{l^2}\left(l\frac{\partial v}{\partial \tau} + 4v^2\right)\right.$$
$$\left.\times\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2} + \left(\frac{v^2\mathscr{S}^2}{l^2} - \frac{1}{l}\frac{\partial v}{\partial \tau}(1-\mathscr{S}) - \frac{v^2(1-\mathscr{S})^2}{l^2}\right)\frac{\partial^3 \mathscr{P}_l}{\partial \mathscr{S}^3}\right)$$
$$\left.+\frac{1}{l^6}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^4 \mathscr{P}_l}{\partial \mathscr{S}^4}\right)\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S} + \int_{\mathscr{S}}^1\left(\left(\frac{1}{2l^3}\frac{\partial v}{\partial \tau} - \frac{2v^2}{l^4}\right)\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\right.$$
$$\left.\left.\left.+\frac{v^2(1-2\mathscr{S})}{l^4}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2} + \frac{1}{l^6}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial^3 \mathscr{P}_l}{\partial \mathscr{S}^3}\right)\mathrm{d}\mathscr{S}\right]\right\}\mathrm{d}\mathscr{S}, \tag{53}$$

$$d_{ijkl} = \int_0^1 \mathscr{P}_i\left\{\frac{2v}{l^3}\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}} + \frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\int_0^{\mathscr{S}}\left[\frac{-2v\mathscr{S}}{l^3}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\right.\right.$$
$$\left.-\frac{2v}{l^3}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}} - \frac{2v}{l^3}\frac{\partial \mathscr{P}_k}{\partial S}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2}\right]\mathrm{d}\mathscr{S}$$

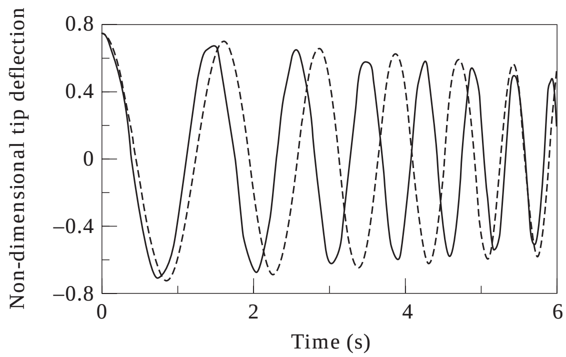

Figure 9. Test Case 1: constant velocity retraction $V_0 = -0{\cdot}1145$ m/s; ——, non-linear; ---, linear.

$$-\frac{\partial^2 \mathscr{P}_j}{\partial \mathscr{S}^2}\left[\int_{\mathscr{S}}^{1}\int_0^{\mathscr{S}}\left(\frac{-2v\mathscr{S}}{l^3}\frac{\partial^2 \mathscr{P}_k}{\partial \mathscr{S}^2}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}} - \frac{2v}{l^3}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\right.\right.$$

$$\left.\left. - \frac{2v}{l^3}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2}\right)\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S} + \int_{\mathscr{S}}^{1}\frac{2v}{l^3}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\,\mathrm{d}\mathscr{S}\right]\Bigg\}\,\mathrm{d}\mathscr{S} \tag{54}$$

and

$$e_{ijkl} = \int_0^1 \mathscr{P}_i\left(\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\int_0^{\mathscr{S}}\frac{1}{l^2}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\,\mathrm{d}\mathscr{S} - \frac{\partial^2 \mathscr{P}_j}{\partial \mathscr{S}^2}\int_{\mathscr{S}}^{1}\int_0^{\mathscr{S}}\frac{1}{l^2}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{S}\right)\mathrm{d}\mathscr{S} \tag{55}$$

Although the coefficients a_{ij}, b_{ij}, c_{ijkl}, d_{ijkl} and e_{ijkl} are time dependent, one may take advantage of the fixed domain by using the space dependent eigenfunctions of the beam. This decreases the computation of these coefficients significantly.

3.3. TIME INTEGRATION AND RESULTS

Following the discussion in section 2, one needs to define a series of comparison functions satisfying all the boundary conditions.

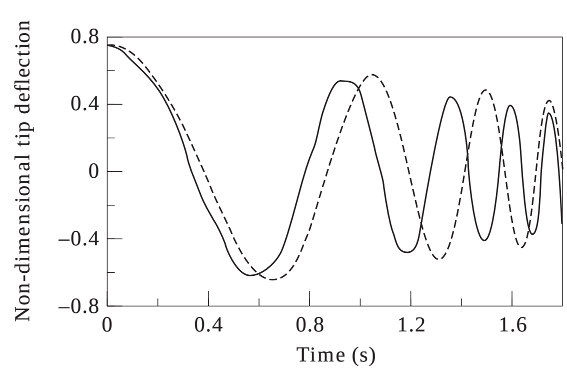

Figure 10. Test Case 1: constant velocity retraction $V_0 = -0{\cdot}5725$ m/s; ——, non-linear; ---, linear.

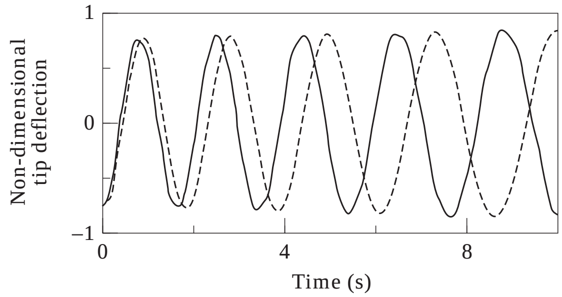

Figure 11. Test Case 2: constant velocity extrusion $V_0 = 0{\cdot}041$ m/s; ——, non-linear; ---, linear.

To study the response of the non-linear system, two types of boundary conditions are examined: the cantilever i.e., a sliding beam with fixed–free boundary conditions and a sliding beam with clamped–clamped boundary condition where the sliding beam emerges from a fixed channel and at the other end is attached to a moving channel with the same prescribed motion.

3.3.1. *Sliding cantilever*

In this case one uses the same ortho-normal eigenfunctions as given in equation (30). Substituting these eigenfunctions into relations (51–55) and using the MAPLE symbolic program, the desired system matrices in terms of the prescribed motion of the beam is computed.

With the discrete equations derived, the set of ODEs is integrated and the transient response is obtained.

The following examples illustrate the difference in the response of the linear and non-linear systems. No physical damping is included in the system equations.

3.3.2. *Spaghetti and reverse spaghetti problems*: *quadratic sliding motion*

A quadratic sliding motion as given in equation (32) is used for the prescribed sliding motion of the beam. In the test case 1, a constant velocity retraction with the initial length $L_0 = 1{\cdot}5$ m and velocity $V_0 = -0{\cdot}1145$ m/s is considered.

Figure 9 shows the difference between the linear and the nonlinear time histories of the non-dimensional tip deflection η of the sliding beam. The differences in frequencies and amplitudes of oscillations become noticeable when the sliding velocity of the beam is increased (see Figure 10).

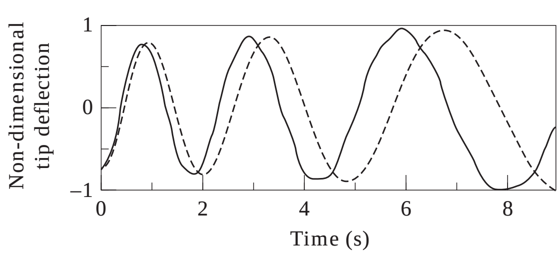

Figure 12. Test Case 2: constant velocity extrusion $V_0 = 0{\cdot}123$ m/s; ——, non-linear; ---, linear.

In test case 2, a constant velocity extrusion with the same initial length and at three different sliding speeds $V_0 = 0{\cdot}041$, $V_0 = 0{\cdot}123$ and $0{\cdot}205$ m/s is examined. The effect of including the non-linear terms as well as increasing the extrusion velocity can be seen in the linear and the non-linear solutions of the system (see Figure 11, 12 and 13).

It is interesting to add a constant acceleration term to the previous reverse spaghetti problem. In this case two different prescribed velocity and acceleration terms for the sliding motion of the beam are considered as follows: $V_0 = 0{\cdot}004$ m/s, $a_0 = 0{\cdot}0075$ m/s^2 and $V_0 = 0{\cdot}008$ m/s, $a_0 = 0{\cdot}015$ m/s^2. The difference between the linear and the non-linear solutions, both in amplitude and frequency of oscillations, will be noticed in Figures 14 and 15.

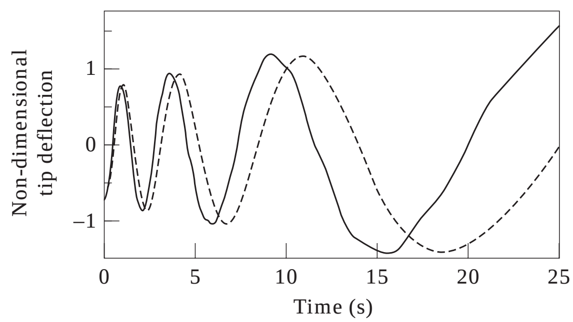

Figure 13. Test Case 2: constant velocity extrusion $V_0 = 0{\cdot}205$ m/s; ——, non-linear; ---, linear.

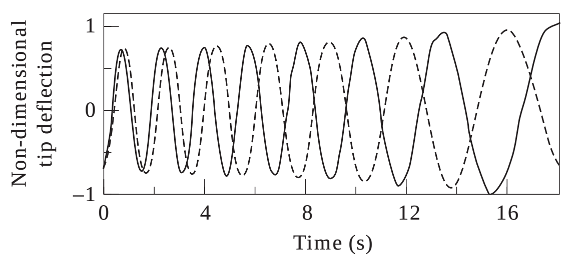

Figure 14. Test Case 4: constant acceleration extrusion $V_0 = 0{\cdot}004$ m/s, $a_0 = 0{\cdot}0075$ m/s; ——, non-linear; ---, linear.

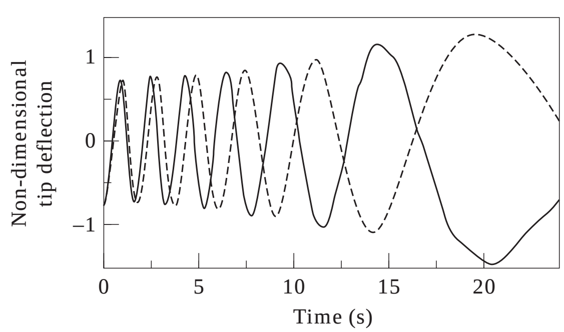

Figure 15. Test Case 4: constant acceleration extrusion $V_0 = 0{\cdot}008$ m/s, $a_0 = 0{\cdot}015$ m/s^2; ——, non-linear; ---, linear.

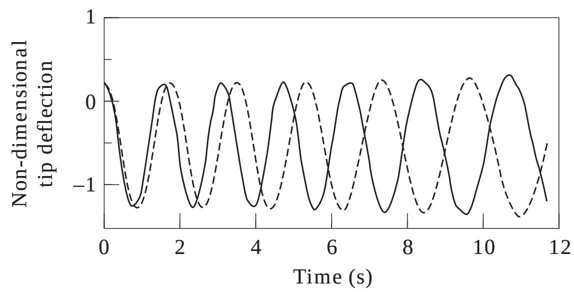

Figure 16. Test Case 5: low frequency extrusion $c_0 = 0{\cdot}7$ m; ——, non-linear; ---, linear.

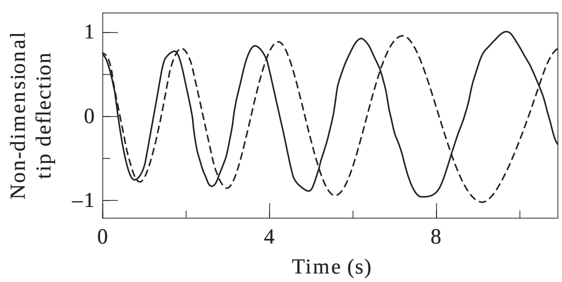

Figure 17. Test Case 5: low frequency extrusion $c_0 = 2{\cdot}1$ m; ——, non-linear; ---, linear.

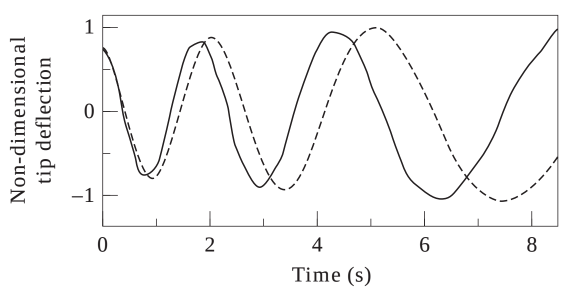

Figure 18. Test Case 5: low frequency extrusion $c_0 = 3{\cdot}5$ m; ——, non-linear; ---, linear.

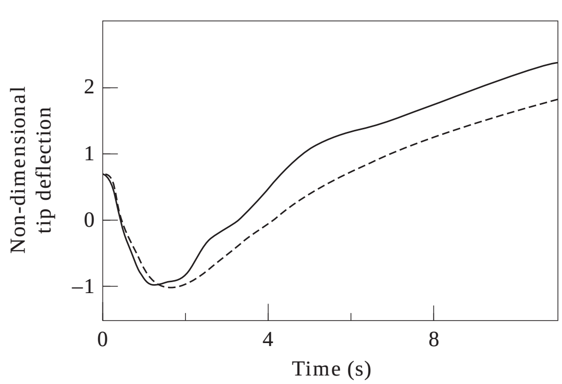

Figure 19. Test Case 6: high frequency extrusion $c_0 = 0{\cdot}7$ m; ——, non-linear; ---, linear.

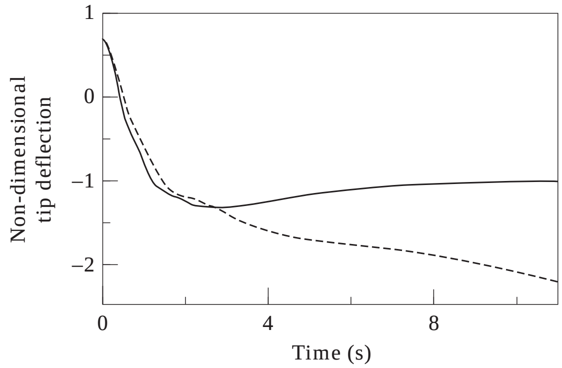

Figure 20. Test Case 6: high frequency extrusion $c_0 = 1{\cdot}05$ m; ——, non-linear; ---, linear.

3.3.3. *Spaghetti and reverse spaghetti problems*: *repositional motion*

The variable length of the beam is given in equation (33). In test case 5, a low frequency extrusion $L_0 = 1{\cdot}5$ m, $c_0 = 0{\cdot}7$ m and $t_0 = 20{\cdot}8$ s is considered. Figure 16 shows the response of the system using the linear and the non-linear equations. The effect of increasing the prescribed velocity of the sliding and the axial oscillation of the beam can be seen in Figures 17 and 18.

In test case 6, the high frequency oscillation reverse spaghetti problem is examined. Increasing the frequency drastically changes the response of the system. In this case $t_0 = 0{\cdot}875$ s in two different prescribed velocities where $c_0 = 0{\cdot}7$ m and $c_0 = 1{\cdot}05$ m is considered. In this case one may notice the considerable difference between the linear response and non-linear response of the system where increasing the velocity leads to unstable solution for the linear case while the non-linear solution is bounded (see Figures 19 and 20).

In test case 7, the high frequency spaghetti problem with initial length $L_0 = 3$ m and $c_0 = -0{\cdot}7$ m is considered. Figure 21 shows the transient response of the system in two different, linear and non-linear, cases. Increasing the retraction velocity has also a significant effect on the system (see Figure 22).

3.3.4. *Reverse spaghetti problem*: *high frequency perturbations*

In this case the constant velocity extrusion is modified by introducing an axial perturbation as given in equation (34).

Here one considers the following parameters to study the effects of non-linear terms in the response of the system: $L_0 = 1{\cdot}5$ m, $\varepsilon = 0{\cdot}0133$ and $\hat{\omega} = 11\omega_1$ where ω_1 is the instantaneous fundamental natural frequency of the sliding beam. In this test physical damping as defined in section 2 is concluded. Figures 23 and 24 show the transient response of the system and the difference between the linear and the non-linear solutions for the two extruding velocities $V_0 = 0{\cdot}1145$ m/s and $V_0 = 0{\cdot}3435$ m/s.

3.3.5. *Sliding clamped–clamped*

For the sliding beam with clamped–clamped boundary condition, one uses the ortho-normal eigenfunctions of the stationary clamped–clamped beam given by

$$\mathscr{P}_i = \cosh\beta_i\eta - \cos\beta_i\eta - \frac{\cos\beta_i - \cosh\beta_i}{\sin\beta_i - \sinh\beta_i}(\sinh\beta_i\eta - \sin\beta_i\eta), \tag{56}$$

where the eigenvalues β_i are the roots of the characteristic equation

$$\cos \beta_i \cosh \beta_i - 1 = 0. \tag{57}$$

Using these modal functions one calculates the system matrices as functions of the time varying length and the sliding velocity of the beam.

Figure 25 shows the non-dimensional mid-span deflection of the beam with initial length $L_0 = 1{\cdot}5$ m extruding with constant velocity $V_0 = 0{\cdot}5725$ m/s.

In another case, one considers the variable length of the beam as defined in equation (33). For the reverse spaghetti problem, the parameters are chosen: $L_0 = 1{\cdot}5\ m$, $c_0 = 0{\cdot}7$ m and $t_0 = 0{\cdot}875$ s. The response of the system is give in Figure 26. Increasing the velocity leads to an unstable solution for the linear system (see Figure 27).

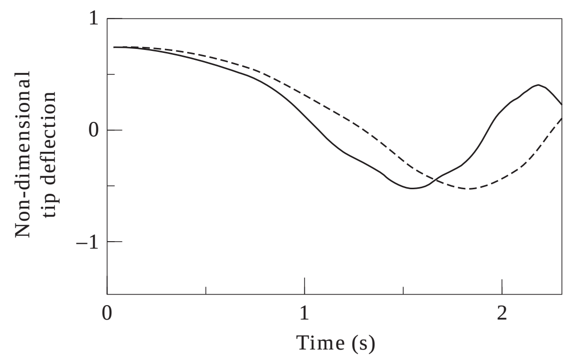

Figure 21. Test Case 7: high frequency retraction $c_0 = -0{\cdot}7$ m; ——, non-linear; ---, linear.

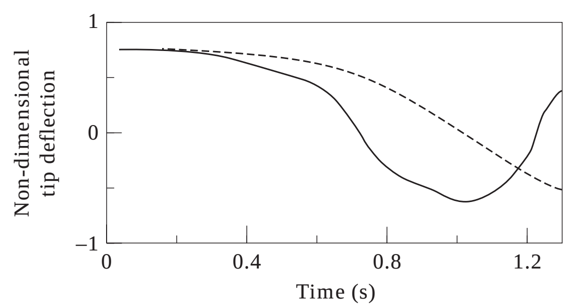

Figure 22. Test Case 7: high frequency retraction $c_0 = -2{\cdot}1$ m; ——, non-linear; ---, linear.

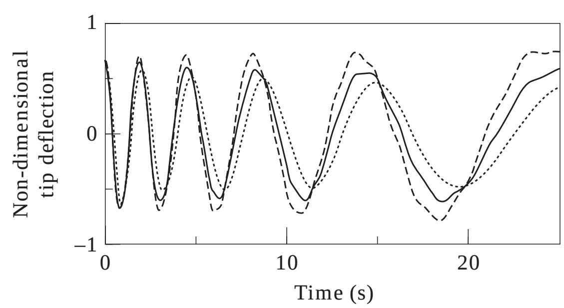

Figure 23. Test Case 8: high frequency perturbation extrusion $V_0 = 0{\cdot}1145$ m/s; ——, non-linear perturbed; ---, non-linear unperturbed.

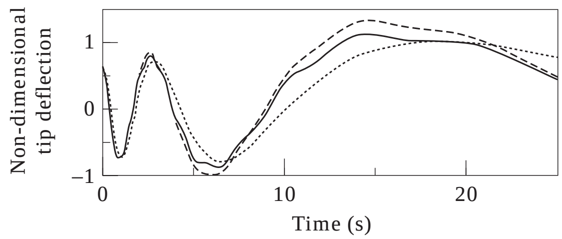

Figure 24. Test Case 8: high frequency perturbation extrusion $V_0 = 0{\cdot}3435$ m/s; ——, non-linear perturbed; ---, non-linear unperturbed.

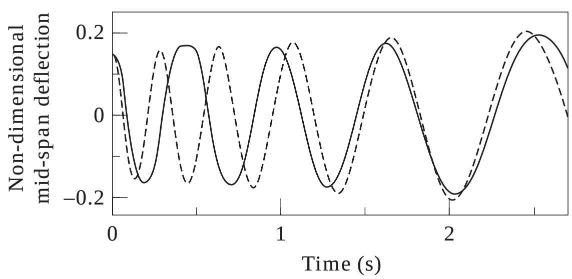

Figure 25. Test Case 9: constant velcoity extrusion $V_0 = 0{\cdot}5725$ m/s; ——, non-linear; ---, linear.

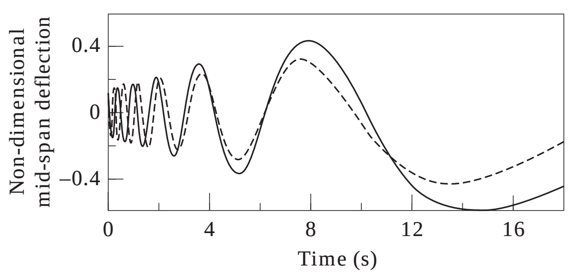

Figure 26. Test Case 10: high frequency extrusion $c_0 = 0{\cdot}7$ m; ——, non-linear; ---, linear.

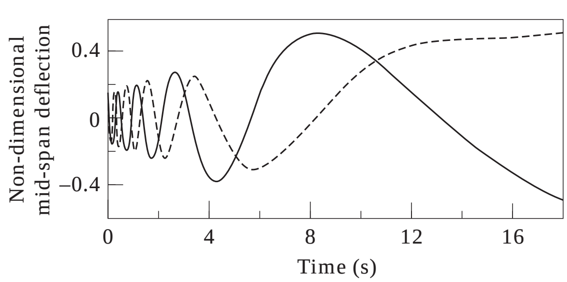

Figure 27. Test Case 10: high frequency extrusion $c_0 = 0{\cdot}931$ m; ——, non-linear; ---, linear.

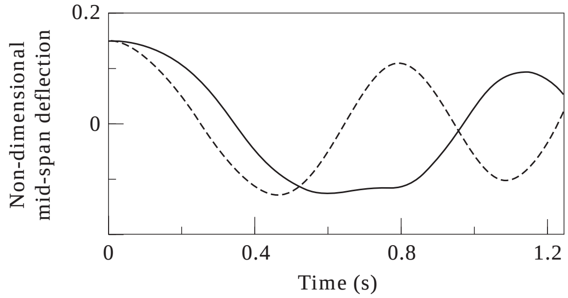

Figure 28. Test Case 11: high frequency retraction $c_0 = -0{\cdot}7$ m; ——, non-linear; ---, linear.

In test case 11, the high frequency retraction case with the initial length of the beam $L_0 = 3{\cdot}0$ m is examined (see Figure 28). In this test physical damping as defined in section 2 is included.

4. NON-LINEAR AXIALLY INEXTENSIBLE FLEXIBLE SLIDING BEAMS IN UNIFORM GRAVITATIONAL FIELD AND IN THE FIXED DOMAIN

Additional terms due to gravity may be obtained as outlined in part I and in the fixed domain they may be expressed as

$$\frac{\rho A g}{l}\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\left[1+\frac{1}{2l^2}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right]-\frac{\rho A g}{l}(1-\mathscr{S})\left[1+\frac{3}{2l^2}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\right]\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}.$$

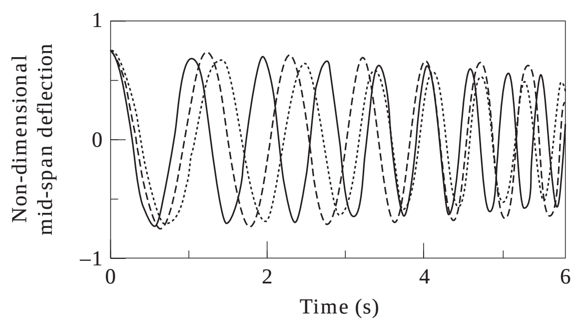

Figure 29. Spaghetti problem: uniform gravitational field, constant velocity retraction $V_0 = -0{\cdot}1145$ m/s; ——, non-linear gravity; ---, linear–gravity.

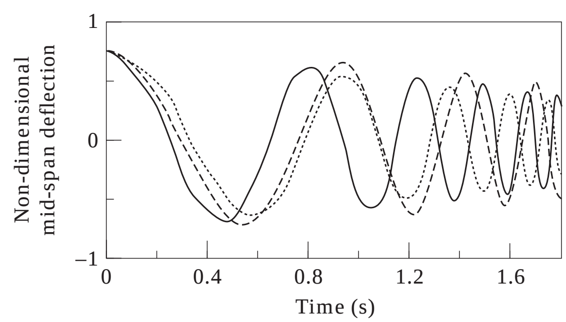

Figure 30. Spaghetti problem: uniform gravitational field, constant velocity retraction $V_0 = -0{\cdot}5725$ m/s; ——, non-linear gravity; ---, linear–gravity; . . . , non-linear–no gravity.

Thus the additional linear and non-linear terms take the form

$$L_1^G = \frac{\rho A g}{l}\frac{\partial \hat{x}_2}{\partial \mathscr{S}} - \frac{\rho A g}{l}(1-\mathscr{S})\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2} \tag{58}$$

and

$$N_1^G = \frac{\rho A g}{2l^3}\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^3 - \frac{3\rho A g}{2l^3}(1-\mathscr{S})\left(\frac{\partial \hat{x}_2}{\partial \mathscr{S}}\right)^2\frac{\partial^2 \hat{x}_2}{\partial \mathscr{S}^2}. \tag{59}$$

To express the above additional terms in dimensionless form, it is required to define a new non-dimensional term as

$$\mathscr{G} = (\rho A g / EI) L_0^3. \tag{60}$$

Therefore, dimensionless additional linear and non-linear terms become

$$\mathscr{L}_1^G(\eta) = \frac{\mathscr{G}}{l}\frac{\partial \eta}{\partial \mathscr{S}} - \frac{\mathscr{G}}{l}(1-\mathscr{S})\frac{\partial^2 \eta}{\partial \mathscr{S}^2} \tag{61}$$

and

$$\mathscr{N}_1^G(\eta) = \frac{\mathscr{G}}{2l^3}\left(\frac{\partial \eta}{\partial \mathscr{S}}\right)^3 - \frac{3\mathscr{G}}{2l^3}(1-\mathscr{S})\left(\frac{\partial \eta}{\partial \mathscr{S}}\right)^2\frac{\partial^2 \eta}{\partial \mathscr{S}^2}. \tag{62}$$

When transforming the PDE to ODEs, the above additional terms introduce additional linear and non-linear stiffness matrices derived as

$$b_{ij}^G = \int_0^1 \mathscr{P}_i\left[\frac{\mathscr{G}}{l}\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}} - \frac{\mathscr{G}}{l}(1-\mathscr{S})\frac{\partial^2 \mathscr{P}_j}{\partial \mathscr{S}^2}\right]\mathrm{d}\mathscr{S} \tag{63}$$

and

$$c_{ijkl}^G = \int_0^1 \mathscr{P}_i\frac{\partial \mathscr{P}_j}{\partial \mathscr{S}}\frac{\partial \mathscr{P}_k}{\partial \mathscr{S}}\left[\frac{\mathscr{G}}{2l^3}\frac{\partial \mathscr{P}_l}{\partial \mathscr{S}} - \frac{3\mathscr{G}}{2l^3}(1-\mathscr{S})\frac{\partial^2 \mathscr{P}_l}{\partial \mathscr{S}^2}\right]\mathrm{d}\mathscr{S}. \tag{64}$$

With the inclusion of these stiffness matrices in the governing equations of motion obtained in section 2, one may study the transient response of the axially rigid sliding beams in a uniform gravitation field.

Here test case 1 under a uniform gravitational field is considered. Figures 29 and 30 show the necessity of including the non-linear terms in the solution as well as the effects of uniform gravity field in the response of the system.

5. CONCLUSIONS

In this paper the transient response of axially inextensible sliding beams has been computed. One started with the linear case and transformed the governing equation of motion to the fixed domain. Subsequently Galerkin's method with space dependent modal functions was used to obtain the system discrete set of ODE's. Numerical solutions of the ODE's were in good agreement with the simulation and experimental work reported in the literature [7, 8].

The analysis was further extended to the non-linear case and system equation obtained in part I transformed to the fixed domain. Again using Galerkin's method one obtained a set of non-linear ODE's which upon solution provided the response of the system. Several illustrative cases were considered and the results exposed the differences between the linear and the non-linear solutions to these problems. Also, the effect of uniform gravitational field on the response of the system was studied.

In this paper the effect of axial flexibility which can have a crucial role on the response of the system was neglected. This effect has not been considered by other researchers since the large difference between the axial and flexural rigidities of slender beams would indicate that the axial flexibility has a negligible effect on the motion of the system. In reference [13] the finite element method as used for the solution of axially extensible sliding beams.

ACKNOWLEDGMENTS

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